Advanced Operations Management

Fall 2006, Professor Eckstein

Solutions to Homework 1

1. We have

$$\begin{aligned} D &= 4,000 \text{ gallons/month} = 48,000 \text{ gallons/year} \\ h &= 0.30 \frac{\$}{\text{gallon year}} = 0.30 \frac{\$}{\text{gallon year}} \cdot \frac{1}{12} \frac{\text{year}}{\text{month}} = 0.025 \frac{\$}{\text{gallon month}} \\ K &= \$50. \end{aligned}$$

Therefore

$$q^* = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \cdot 50 \cdot 4000}{0.025}} = 4000 \text{ gallons.}$$

Most errors on this problem were caused by being inconsistent between months and years in the units for D and h. Some people were inconsistent in the calculation they wrote down, but then somehow got the right answer.

Next, $D/q = 4000/4000 = 1$ order/month $= 12$ orders/year, so there will be one month between orders. This answers parts (a), (b), and (c).

For part (e): if $L = 2$ weeks $= (2/52)$ year, then

$$LD = (2/52)(48,000) \approx 1846 \text{ gallons} < q^*,$$

and orders should be placed when 1846 gallons are in stock. If $L = 10$ weeks $= (10/52)$ year, then

$$LD = (10/52)(48,000) \approx 9231 \text{ gallons} > q^*.$$

We order when the stock reaches $LD \bmod q^* = 9231 \bmod 4000 = 1231$ (the remainder when 9231 is divided by 4000).

2. This problem resembles the bus example on page 112. Instead of an inventory of pizzas suddenly arriving and being gradually consumed, we gradually build up an inventory of baked pizzas, which are suddenly removed by the truck. If q is the number of pizzas on a truck, our average inventory is still $q/2$, and in every other respect the model is identical to the standard EOQ setup. So, for part (a), was can use the standard EOQ formula with

$$D = 30 \frac{\text{calls}}{\text{hour}} \qquad K = \$10 \qquad h = 12 \frac{\$}{\text{pizza hour}}.$$

The EOQ formula gives that a truck should carry

$$q^* = \sqrt{\frac{2(10)(30)}{12}} \approx 7.07 \text{ pizzas.}$$

The number of trucks sent out per hour should thus be

$$\frac{30\ \frac{\text{pizzas}}{\text{hour}}}{7.07\ \frac{\text{pizzas}}{\text{truck}}} \approx 4.24\ \frac{\text{trucks}}{\text{hour}}.$$

In reality, we can't put 7.07 pizzas on a truck; we'd probably just send out one truck for every 7 pizzas produced.

For part (b), we constrain the number of pizzas per truck to be at most 5. Thus, we have the problem

$$\begin{aligned} \min \quad & f(q) = \frac{KD}{q} + \frac{hq}{2} \\ \text{ST} \quad & 0 < q \leq 5 \end{aligned}$$

Setting the derivative to zero yields $q^* \approx 7.07$, as in part (a). Note that $f''(q) = 2KD/q^3 > 0$ for all $q > 0$, so the problem is convex, and we can just choose the closest endpoint to q^*, namely $q = 5$. In that case, we send out

$$\frac{30\ \frac{\text{pizzas}}{\text{hour}}}{5\ \frac{\text{pizzas}}{\text{truck}}} \approx 6\ \frac{\text{trucks}}{\text{hour}}.$$

3. There was massive confusion on this problem. First, by "cost of goods sold during the year", the book meant just pD. Instead, I think all of you just used the total cost formula from the EOQ model. I already knew the definition of turnover ratio, so I didn't realize how confusing the book's definition was when I read the problem. Second, you were supposed to substitute $q = \sqrt{2KD/h}$ into the turnover formula and express it in terms of K, D, and h. But a typo in part (b) in the book also made that unclear. So, I decided not to grade this problem, and just give everybody full credit. But here is the solution that you should have had if the problem had been clearer:

   For part (a), we first note that the numerator is pD, the "accounting" cost of goods sold. Second, if the order amount q is given by the EOQ formula, the average value of goods on hand is

$$p\frac{q}{2} = \frac{p}{2}\sqrt{\frac{2KD}{h}} = p\sqrt{\frac{KD}{2h}}.$$

   Therefore,

$$TR = \frac{pD}{p\sqrt{\frac{KD}{2h}}} = \frac{D}{\sqrt{\frac{KD}{2h}}} = \sqrt{\frac{2Dh}{K}}.$$

   For part (b), note that an increase in D increases the turnover ratio. Basically, that is because the value of goods sold increases proportionally to D, but the value on hand, like the EOQ order amount, increases only proportionally to $\sqrt{D}$.

   Finally, an increase in K decreases turnover. Larger K means less frequent orders, more stock on hand, and lower turnover.

4. (a) Consider district one. The time elapsing between cleanings is $1/p$ weeks. At the time of a cleaning, the amount of litter accumulated is

$$\left(R_1 \, \frac{\text{tons}}{\text{week}}\right)\left(\frac{1}{p}\,\text{weeks}\right) = \frac{R_1}{p}\,\text{tons}.$$

If cleanings remove all the accumulated litter, and litter accumulates at a steady rate, then — just as in the EOQ model or the bus problem on page 112 — the average amount of litter in the district is just half the amount at the time of cleaning, namely $R_1/2p$. For district two, the logic is identical, except we use R_2 instead of R_1, and $k - p$ instead of p.

Quite a few people wrote that if litter accumulates at rate R_1, then the "average rate of accumulation" must be $R_1/2$. I couldn't make any sense of that.

(b) The average amount of litter on the city's streets is just the sum of the amounts in the two districts. The number of cleanings per week in district one must be between 0 and k. We thus obtain the problem

$$\begin{array}{ll} \min & \dfrac{R_1}{2p} + \dfrac{R_2}{2(k-p)} \\ \text{ST} & 0 \le p \le k. \end{array}$$

We next try to set the derivative of the objective function to zero. Much as in the EOQ model, the derivative of the first term is $-R_1/2p^2$. For the second term, you can use Table 2 on page 3 of the text (especially line 9, which says that $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$) to ascertain that

$$\left(\frac{d}{dp}\right)\frac{R_2}{2(k-p)} = \frac{-R_2}{2(k-p)^2}(-1) = \frac{R_2}{2(k-p)^2}.$$

Note that lines 9-12 of Table 2 are all special cases of the classic *chain rule*, which says that $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$. Returning to the problem at hand, setting the derivative of the entire objective to zero yields the equation

$$-\frac{R_1}{2p^2} + \frac{R_2}{2(k-p)^2} = 0,$$

which rearranges to

$$\frac{R_2}{2(k-p)^2} = \frac{R_1}{2p^2}.$$

Canceling the factor of 2 and cross-multiplying yields

$$R_2 p^2 = R_1(k-p)^2$$

(at this point, some people did unnecessarily complicated things). Taking square roots then produces

$$\sqrt{R_2}\,p = \sqrt{R_1}(k-p).$$

Adding $\sqrt{R_1}\,p$ to both sides then gives

$$\left(\sqrt{R_1} + \sqrt{R_2}\right) p = k\sqrt{R_1},$$

and dividing through by $\sqrt{R_1} + \sqrt{R_2}$ results in

$$p = \frac{k\sqrt{R_1}}{\sqrt{R_1} + \sqrt{R_2}}.$$

For nonnegative R_1 and R_2, this value must be between 0 and k. If you graph the objective function, you will see that it is $+\infty$ for $p = 0$ and $p = k$, so those cannot be solutions (alternatively, you can take second derivatives, and note that the problem is convex). Thus, the formula for p above gives the solution.

(c) Plugging $R_1 = 2000$, $R_2 = 1000$, and $k = 1$ into the formula for p yields

$$p = \frac{\sqrt{2000}}{\sqrt{2000} + \sqrt{1000}} = \frac{\sqrt{2}}{\sqrt{2} + 1} \approx 0.586.$$

So we clean district one approximately 0.586 times per week, and thus we clean district two $1 - p = 1/(\sqrt{2} + 1) \approx 0.414$ times per week.

Intuition might suggest that since litter accumulates twice as fast in district one, it should be cleaned twice as often as district two, but in fact it should be cleaned $p/(1 - p) = \sqrt{2}$ times as often, that is, about 41.4% more than district two.