Convex Analysis and Optimization 16:711:558

Fall 2011 Rutgers University Prof. Eckstein

Homework 6

Due *Monday*, November 21

Note that we meet Monday instead of Wednesday on Thanksgiving week.

1. Given some closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and nonempty closed convex cone $K \subseteq \mathbb{R}^n$, consider the conic optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{ST} & x \in K. \end{array} \tag{1}$$

   (a) Using the Fenchel-like duality scheme from the April 8 class, show that the dual problem to (1) may be expressed as

$$\begin{array}{ll} \max & -\widehat{f}(\lambda) \\ \text{ST} & \lambda \in -K^*, \end{array} \tag{2}$$

   (b) In the setup of question 1a, show that if $\operatorname{ri} \operatorname{dom} f \cap \operatorname{ri} K \neq 0$ and (1) has a finite optimal solution, then strong duality holds between (1) and (2):

$$\inf \{ f(x) \mid x \in K \} = \sup \left\{ -\widehat{f}(\lambda) \mid \lambda \in -K^* \right\}.$$

   (c) Now using the Rockafellar conjugate duality scheme, show that that if we set

$$F(x,u) = \begin{cases} f(x), & \text{if } x - u \in K \\ +\infty, & \text{if } x - u \notin K, \end{cases}$$

   we obtain the same dual problem (2). What is the Lagrangian $L(x, \lambda)$ in this case?

   (d) Show that in the setup of question 1c, the following additional conditions are sufficient to assure strong duality between the problems (1) and (2):

   - There is some $\bar{\lambda} \in -K^*$ with $\widehat{f}(\bar{\lambda}) < \infty$
   - The function f is coercive; that is, if $\{z^k\} \subset \mathbb{R}^n$ is a sequence with $\|z^k\| \to \infty$, then $f(z^k) \to +\infty$.

   Note: there are many possible alternative sufficient conditions.

2. In this problem, you will show how to recover the Fenchel duality scheme from the Rockafellar conjugate scheme, and derive the proper form of the Lagrangian for the Fenchel problem. Consider two functions closed proper convex functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, and an $m \times n$ matrix M. Consider the standard Fenchel primal problem

$$\min_{x \in \mathbb{R}^n} f(x) + g(Ax). \tag{3}$$

(a) Show that if we set $F(x, u) = f(x) + g(Ax + u)$, then F is closed and convex, and we obtain the Lagrangian

$$L(x, \lambda) = f(x) + \langle Ax, \lambda \rangle - \widehat{g}(\lambda),$$

with the convention that if both $f(x)$ and $\widehat{g}(\lambda)$ are $+\infty$, then $L(x, \lambda) = +\infty$.

(b) Show that Rockafellar conjugate duality scheme assigns to (3) the dual problem

$$\max_{\lambda \in \mathbb{R}^m} -\widehat{f}(-A^\top \lambda) - \widehat{g}(\lambda), \tag{4}$$

that is, the same dual as you obtain in the Fenchel approach.