Convex Analysis and Optimization 16:711:558

Fall 2013 Rutgers University Prof. Eckstein

Homework 1

Due Thursday, September 19

1. *Affine images and preimages of convex sets.* Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, and $C \subseteq \mathbb{R}^n$, $D \subseteq \mathbb{R}^m$ be convex sets. Show that following sets are convex:

   (a) The image of C under the affine map $x \mapsto Ax + b$, that is,

   $$AC + b = \{Ax + b \mid x \in C\} \subseteq \mathbb{R}^m.$$

   (b) The "preimage" of D under the same mapping, that is, $\{x \in \mathbb{R}^n \mid Ax + b \in D\}$.

2. *Affine functions.* Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ (*not* taking $\pm\infty$ values) always obeys the convex function relation at equality, that is,

   $$f\big(\alpha x + (1-\alpha)y\big) = \alpha f(x) + (1-\alpha) f(y) \qquad \forall\, x, y \in \mathbb{R}^n,\ \alpha \in [0,1]. \tag{1}$$

   Show that if (1) holds as stated for all $\alpha \in [0,1]$, it in fact holds for all $\alpha \in \mathbb{R}$. Then show that any f for which (1) holds must be of the form $f(x) = \langle a, x\rangle + b$ for $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ (that is, f is an inner product with some fixed vector, plus a constant). Conversely, also show that any function of this form has the property (1). *Hint:* given f satisfying the condition above, show that $g : x \mapsto f(x) - f(0)$ is linear; you may then use (without proof, although the proof is very easy) that a linear function $g : \mathbb{R}^n \to \mathbb{R}$ must be of the form $x \mapsto \langle a, x\rangle$ for some $a \in \mathbb{R}^n$.

3. *Convex hulls.* Show that for any set $X \subseteq \mathbb{R}^n$, the convex hull $\operatorname{conv}(X)$ of X (the intersection of all convex sets containing X) is equal to the set of all convex combinations of points in X.

4. *Affine sets and hulls.* Note: the scalars α_i in this problem may take negative values.

   (a) The textbook defines a set $X \subseteq \mathbb{R}^n$ as being *affine* if it is of the form $S + x = \{x + s \mid s \in S\}$ for some $x \in \mathbb{R}^n$ and linear subspace S of $\mathbb{R}^n$. Show that X is affine according to this definition if and only if X has the property

   $$\left.\begin{array}{l} x^1, \ldots, x^m \in X \\ \alpha_1, \ldots, \alpha_m \in \mathbb{R} \\ \sum_{i=1}^m \alpha_i = 1 \end{array}\right\} \quad \Rightarrow \quad \sum_{i=1}^m \alpha_i x^i \in X$$

   Hint: for the "if" direction, take any $x \in X$ and show that the set $S = X - x = \{x' - x \mid x' \in X\}$ is a linear subspace.

(b) In the text, the *affine hull* $\operatorname{aff}(Y)$ of a set Y is defined to be the intersection of all affine sets containing Y. Show that

$$\operatorname{aff} Y = \left\{ \sum_{i=1}^{m} \alpha_i y^i \;\middle|\; m > 0,\ y^1, \ldots, y^m \in Y,\ \alpha_1, \ldots, \alpha_m \in \mathbb{R},\ \sum_{i=1}^{m} \alpha_i = 1 \right\},$$

that is, $\operatorname{aff}(Y)$ is the set of all affine combinations of points from Y.

5. *Calculus/analysis applications: arithmetic/geometric means.* Exercise 1.11 on page 73 of the text.