Special Topics in Management Science 26:711:685

Convex Analysis and Optimization

Fall 2013 Rutgers University Prof. Eckstein

Homework 2

Due Thursday, September 26

1. Prove the following fact, used in class: if $f : \mathbb{R}^n \to (-\infty, +\infty]$ is closed, then all its level sets $L_\gamma = \{x \in \mathbb{R}^n \mid f(x) \leq \gamma\}$, $\gamma \in \mathbb{R}$ are closed.

2. Suppose that $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex function and $x \in \operatorname{dom} f$. Show that for any $d \in \mathbb{R}^n$ the function $g_d : (0, \infty) \to (-\infty, +\infty]$ defined by
$$g_d(\alpha) = \frac{f(x + \alpha d) - f(x)}{\alpha}$$
is nondecreasing.

3. Nonconvex Projections (similar to exercise 2.11 in the text): Let $C \subset \mathbb{R}^n$ be a nonempty closed set (but possibly not convex), and consider any point $x \in \mathbb{R}^n$.
   (a) Show that the function $g(w) = \|w - x\|$ must have a nonempty, compact set of minima over C. Denote this set by $P_C(x)$.
   (b) Show that $d_C(x) = \inf_{w \in C} \|w - x\|$ is an everywhere finite-valued and continuous function of $x \in \mathbb{R}^n$.
   (c) Give an example showing that if C is not convex, d_C need not be convex.

4. Given a set $X \subseteq \mathbb{R}^n$, its *indicator function* is the function $\delta_X : \mathbb{R}^n \to (\infty, +\infty]$ given by
$$\delta_X(x) = \begin{cases} 0, & \text{if } x \in X \\ +\infty, & \text{if } x \notin X. \end{cases}$$
   (a) Show that if X is a closed set, δ_X is a closed function.
   (b) Show that if X is a convex set, δ_X is a convex function.