Special Topics in Management Science 26:711:685

Convex Analysis and Optimization

Fall 2013 Rutgers University Prof. Eckstein

Homework 3

Due Thursday, October 10

1. Problems 3.4(a)-(c) on page 212 of the textbook. Hint: to show $C^* = K$, the simplest general strategy is usually to show that $\langle x, y\rangle \le 0$ for all $x \in C$ and $y \in K$, establishing $K \subseteq C^*$, and then show that if $z \notin K$, then there exists some $x \in C$ with $\langle x, z\rangle > 0$, implying that $z \notin C^*$, and thus $C^* = K$ since $z \notin K$ was arbitrary.

2. *Cone separation.* Suppose $K \subset \mathbb{R}^n$ is a nonempty closed convex cone. Show that if $z \in \mathbb{R}^n$ and $z \notin K$, then there exists $a \in K^*$ with $\langle a, z\rangle > 0$.

3. Show that if $C_1, C_2 \subseteq \mathbb{R}^m$ are closed convex cones, then $(C_1 \cap C_2)^* = \operatorname{cl}(C_1^* + C_2^*)$. Note: this is the main result of problem 3.4(d) in the textbook. Hint: to show that $z \notin \operatorname{cl}(C_1^* + C_2^*)$ implies $z \notin (C_1 \cap C_2)^*$, use problem 2, problem 1(c), and the polar cone theorem.

4. Let A be an $m \times n$ real matrix, and $C \subseteq \mathbb{R}^m$ be a closed convex cone. Define

$$K = \{x \in \mathbb{R}^n \mid Ax \in C\} \qquad\qquad P = \{A^\top y \mid y \in C^*\}.$$

   (a) Show that K is a closed convex cone.

   (b) Show that $K^* = \operatorname{cl} P$. Hint: to show that $z \notin \operatorname{cl} P$ implies $z \notin K^*$, use problem 2.

   (c) Show that $P^* = K$ (by using the polar cone theorem).

5. A cone K is called *self-dual* if $K^* = -K$. Show that the following cones are self-dual:

   (a) The nonnegative orthant $\{x \in \mathbb{R}^n \mid x \ge 0\}$

   (b) The *Lorentz cone* (also called the "ice cream cone") in $\mathbb{R}^{n+1}$, defined as follows:

$$K = \{(x, w) \in \mathbb{R}^n \times \mathbb{R} \mid w \ge \|x\|\}.$$