Special Topics in Management Science 26:711:685

Convex Analysis and Optimization

Fall 2013 Rutgers University Prof. Eckstein

Homework 5

Due Thursday, October 24

Note: since this homework will be considered one of the "take-home exams", you may not collaborate with other students in any way. You may refer to the course textbook, earlier homework assignments and solutions, and your notes from class; you may not refer to other materials. You may treat any result proved in class or on an earlier homework as given.

1. **Normal cones to level sets.** Suppose $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable convex function, and consider the level set $L(0, h) = \{x \in \mathbb{R}^n \mid h(x) \le 0\}$. Assuming that there exists some point $\bar{x} \in \mathbb{R}^n$ with $h(\bar{x}) < 0$, prove that, for any $x \in L(0, h)$, the normal cone $N_{L(0,h)}(x)$ to $L(0, h)$ at x is given by the formula

$$N_{L(0,h)}(x) = \{\alpha \nabla h(x) \mid \alpha \ge 0, \alpha h(x) = 0\} = \begin{cases} \{\alpha \nabla h(x) \mid \alpha \ge 0\}, & \text{if } h(x) = 0 \\ \{0\}, & \text{if } h(x) \ne 0. \end{cases}$$

2. **Optimality conditions for convex problems with "mixed" constraint sets.** Consider an optimization problem of the form

$$\begin{array}{ll} \min & f(x) \\ \text{S.T.} & Ax = b \\ & h_j(x) \le 0 \qquad j = 1, \ldots, r \\ & x \in X, \end{array} \tag{1}$$

where

- $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex function
- A is an $m \times n$ matrix and $b \in \mathbb{R}^m$
- For $j = 1, \ldots, r$, $h_j : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function
- X is a convex set.

Let a_i denote row i of A, $i = 1, \ldots, m$, represented as a column vector. Suppose that there exists a point $\overline{x} \in \mathbb{R}^n$ with the following properties:

- $\overline{x} \in \operatorname{ri} \operatorname{dom} f$
- $A\overline{x} = b$
- For $j = 1, \ldots, r$, $h_j(\overline{x}) < 0$
- $\bar{x} \in \operatorname{ri} X$.

Show that for $x^* \in \mathbb{R}^n$ to be a solution of (1), it is necessary and sufficient that there exist $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^r$ such that

$$\begin{array}{lll} 0 \in \partial f(x^*) + \sum_{i=1}^m \lambda_i^* a_i + \sum_{j=1}^r \mu_j^* \nabla h_j(x^*) + N_X(x^*) & & \sum_{j=1}^r \mu_j^* h_j(x) = 0 \\ Ax^* = b \qquad h_i(x^*) \le 0, \quad i = 1, \ldots, r & & \mu^* \ge 0. \end{array}$$

3. **Optimality conditions for convex cone programming.** Below, suppose $K \subseteq \mathbb{R}^m$ be a nonempty closed convex cone, A is an $m \times n$ matrix, and $b \in \mathbb{R}^m$, and let $Z = \{x \in \mathbb{R}^n \mid Ax - b \in K\}$. Assume that Z is nonempty.

   (a) Show that for any $x \in K$,

   $$F_K(x) = \{z - \alpha x \mid z \in K, \alpha \ge 0\}.$$

   (b) Show that for any $x \in K$,

   $$N_K(x) = \{y \in K^* \mid \langle x, y \rangle = 0\}.$$

   Hint: you may use the results of homework 3, problems 1(c) and 4(c).

   (c) Suppose A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, and let $Z = \{x \in \mathbb{R}^n \mid Ax - b \in K\}$. Show that, for $x \in Z$,

   $$N_Z(x) = \operatorname{cl}\{A^\top \lambda \mid \lambda \in K^*, \langle Ax - b, \lambda \rangle = 0\}. \tag{2}$$

   Hint: you may use the results of homework 3, problem 4.

   (d) Show that $\operatorname{ri} Z \supseteq \{x \in \mathbb{R}^n \mid Ax - b \in \operatorname{ri} K\}$.

   (e) Let $f : \mathbb{R}^n \to (-\infty, +\infty)$ be a convex function, suppose that the cone $A^\top K^* = \{A^\top \lambda \mid \lambda \in K^*\}$ is closed, and consider the problem

   $$\begin{array}{ll} \min & f(x) \\ \text{S.T.} & Ax - b \in K. \end{array} \tag{3}$$

   Further suppose that there exists some point $\bar{x} \in \operatorname{ri} \operatorname{dom} f$ such that $A\bar{x} - b \in \operatorname{ri} K$. Show that, in order for $x^* \in \mathbb{R}^n$ to solve (3), it is necessary and sufficient that there exist some $\lambda^* \in \mathbb{R}^m$ such that

   $$\partial f(x^*) + A^\top \lambda^* \ni 0 \qquad \lambda^* \in K^* \qquad \langle Ax^* - b, \lambda^* \rangle = 0.$$