Special Topics in Management Science 26:711:685

Convex Analysis and Optimization

Fall 2013 Rutgers University Prof. Eckstein

Homework 6

Due Thursday, November 7

1. For each of the following choices of $f : \mathbb{R} \to (-\infty, +\infty]$, compute the convex conjugate function f^*:

   (a) $f(x) = \frac{1}{2}x^2$

   (b) For $a, b \in \mathbb{R}$, $a < b$, setting $f = \delta_{[a,b]}$, that is, $f(x) = 0$ whenever $a \le x \le b$, and otherwise $f(x) = +\infty$

   (c) $f(x) = e^x$ (the standard exponential function).

2. A function $f : \mathbb{R}^n \to (-\infty, +\infty]$ is said to be *positively homogeneous* if $f(0) = 0$ and

$$f(\alpha x) = \alpha f(x)$$

   for all $x \in \mathbb{R}^n$ and scalars $\alpha > 0$. (Note that some definitions omit the condition $f(0) = 0$, which we include here to accord with our notion of a cone as always containing the point 0.)

   (a) For any proper function $f : \mathbb{R}^n \to (-\infty, +\infty]$, show that $\operatorname{epi} f$ is a cone in $\mathbb{R}^{n+1}$ if and only f is positively homogeneous.

   (b) Consider any nonempty set $X \subseteq \mathbb{R}^n$. The *support function* of X is the convex conjugate δ_X^* of the indicator function δ_X (which returns 0 for arguments in X and $+\infty$ otherwise). Show that

$$\delta_X^*(y) = \sup_{x \in X}\{\langle x, y\rangle\},$$

   and that this function is positively homogeneous.

   (c) Show conversely that, given any positively homogeneous function f, its convex conjugate f^* is the indicator function of some closed convex set C.

   (d) Given a cone K, show that $\delta_K^* = \delta_{K^*}$, that is, the conjugate of the indicator function of K is the indicator function of its polar.

3. Given some closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and nonempty closed convex cone $K \subseteq \mathbb{R}^n$, consider the conic optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{ST} & x \in K. \end{array} \tag{1}$$

(a) Show that under the Fenchel duality scheme, the dual problem to (1) may be expressed as

$$\begin{array}{ll} \max & -f^*(\lambda) \\ \text{ST} & \lambda \in -K^*, \end{array} \tag{2}$$

(b) In the setup of question 3a, show that if $\operatorname{ri}\operatorname{dom} f \cap \operatorname{ri} K \neq 0$ and (1) has a finite optimal solution, then strong duality holds between (1) and (2):

$$\inf\{f(x) \mid x \in K\} = \sup\{-f^*(\lambda) \mid \lambda \in -K^*\}.$$

(c) Now using the Rockafellar conjugate duality scheme, show that that if we set

$$F(x,u) = \begin{cases} f(x), & \text{if } x - u \in K \\ +\infty, & \text{if } x - u \notin K, \end{cases}$$

we obtain the same dual problem (2). Give a complete formula for the Lagrangian $L(x, \lambda)$ in this case.