Special Topics in Management Science 26:711:685

Convex Analysis and Optimization

Fall 2013 Rutgers University Prof. Eckstein

Homework 7

Due Thursday, November 21

1. Consider the setup of the last problem on the previous homework, in which we considered some closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and nonempty closed convex cone $K \subseteq \mathbb{R}^n$, and consider the conic optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{ST} & x \in K. \end{array} \tag{1}$$

Suppose, as in the last homework we use the Rockafellar conjugate duality scheme with

$$F(x, u) = \begin{cases} f(x), & \text{if } x - u \in K \\ +\infty, & \text{if } x - u \notin K, \end{cases}$$

which leads to the same dual as obtained by Fenchel duality,

$$\begin{array}{ll} \max & -f^*(\lambda) \\ \text{ST} & \lambda \in -K^*, \end{array} \tag{2}$$

Suppose that the following conditions hold:

- There is some $\bar{\lambda} \in -K^*$ with $f^*(\bar{\lambda}) < \infty$
- The function f is coercive; that is, if $\{z^k\} \subset \mathbb{R}^n$ is a sequence with $\|z^k\| \to \infty$, then $f(z^k) \to +\infty$.

(a) Show that F is a closed proper convex function.

(b) Next, show that the function $\phi(u) = \inf_{x \in \mathbb{R}^n}\{F(x, u)\}$ is closed by establishing the sufficient condition obtained in class, namely that

$$\Pi(\operatorname{epi} F) = \{(u, z) \mid (x, u, z) \in \operatorname{epi} F\}$$

is closed.

(c) Show that asymptotic strong duality holds between (1) and (2), that is, that $\inf_{x \in K} f(x) = \sup_{\lambda \in -K^*} -f^*(\lambda)$. (Note that we did *not* assume have to assume that $\operatorname{ri} \operatorname{dom} f \cap \operatorname{ri} K \neq \emptyset$.)

2. In this problem, you will show how to recover the Fenchel duality scheme from the Rockafellar conjugate scheme, and derive the correct form of the Lagrangian for the Fenchel problem. Consider two closed proper convex functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, and an $m \times n$ matrix A. Consider the standard Fenchel primal problem

$$\min_{x \in \mathbb{R}^n} f(x) + g(Ax), \tag{3}$$

and set $F(x, u) = f(x) + g(Ax + u)$. Show the following:

(a) F is closed proper convex.

(b) The corresponding Lagrangian is

$$L(x,\lambda) = f(x) + \langle Ax, \lambda \rangle - g^*(\lambda),$$

with the convention that if both $f(x)$ and $g^*(\lambda)$ are $+\infty$, then $L(x,\lambda) = +\infty$.

(c) The dual $\max_\lambda \widehat{F}(0,\lambda)$ that we obtain for (3) (or equivalently $\min_x F(x,0)$ is

$$\max_{\lambda \in \mathbb{R}^m} -f^*(-A^\top \lambda) - g^*(\lambda), \tag{4}$$

that is, the same dual as you obtain in the Fenchel approach.

3. Recall that a *monotone operator* on $\mathbb{R}^n$ is a set $T \subset \mathbb{R}^n \times \mathbb{R}^n$ with the property that $\langle x - x', y - y' \rangle \geq 0$ for all $(x, y), (x', y') \in T$, and that such a monotone operator is called *maximal* if it is not a proper subset of any other set T' with this property. Prove the following elementary properties of maximal monotone operators:

   (a) Given any $x \in \mathbb{R}^n$, set $T(x) = \{y \in \mathbb{R}^n \mid (x, y) \in T\}$ must be convex and closed.

   (b) T must be a closed set. That is, if $\{(x^k, y^k)\}$ is any convergent sequence taken from T, its limit (x^∞, y^∞) must also be in T.