Special Topics in Operations Research 16:711:611

Convex Analysis and Optimization

Spring 2009 Rutgers University Prof. Eckstein

Homework 1

Due Wednesday, January 28

1. *Scaling convex sets.* Exercise 1.1 on page 70 of the text (mainly, to prove Proposition 1.2.1(c) in the text).

2. *Affine functions.* Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ (*not* taking $\pm\infty$ values) always obeys the convex function relation at equality, that is,

$$f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha) f(y) \qquad \forall\, x, y \in \mathbb{R}^n. \tag{1}$$

   Show that f must be of the form $f(x) = \langle a, x \rangle + b$ for $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ (that is, f is an inner product with some fixed vector, plus a constant). Conversely, also show that any function of this form has the property (1).

3. *Convex hulls.* Show that for any set $X \subseteq \mathbb{R}^n$, the convex hull $\operatorname{conv}(X)$ of X (the intersection of all convex sets containing X) is equal to the set of all convex combinations of points in X.

4. *Affine sets and hulls.* Note: some of the scalars α_i in this problem may be negative.

   (a) The textbook defines a set $X \subseteq \mathbb{R}^n$ as being *affine* if it is of the form $S + x = \{x + s \mid s \in S\}$ for some $x \in \mathbb{R}^n$ and linear subspace S of $\mathbb{R}^n$. Show that X is affine according to this definition if and only if X has the property

$$\left.\begin{array}{l} x^1, \ldots, x^m \in X \\ \alpha_1, \ldots, \alpha_m \in \mathbb{R} \\ \sum_{i=1}^m \alpha_i = 1 \end{array}\right\} \quad \Rightarrow \quad \sum_{i=1}^m \alpha_i x^i \in X$$

   (b) In the text, the *affine hull* $\operatorname{aff}(Y)$ of a set Y is defined to be the intersection of all affine sets containing Y. Show that

$$\operatorname{aff} Y = \left\{ \sum_{i=1}^m \alpha_i y^i \;\middle|\; m > 0,\ y^1, \ldots, y^m \in Y,\ \alpha_1, \ldots, \alpha_m \in \mathbb{R},\ \sum_{i=1}^m \alpha_i = 1 \right\},$$

   that is, $\operatorname{aff}(Y)$ is the set of all affine combinations of points from Y.

5. *Calculus/analysis applications: arithmetic/geometric means.* Exercise 1.11 on page 73 of the text.