Special Topics in Operations Research 16:711:611

Convex Analysis and Optimization

Spring 2009 Rutgers University Prof. Eckstein

Homework 2

Due Wednesday, February 11

1. Prove the following fact, used in the proof of Proposition 1.5.1(b) in the text: let $x, y \in \mathbb{R}^n$, and let $\{z^k\} \subset \mathbb{R}^n$ be a sequence with $\|z^k\| \to \infty$; then

$$\lim_{k\to\infty} \frac{\|z^k - x\|}{\|z^k - y\|} = 1.$$

Hints: consider the square of the ratio above, instead of the ratio. Expand the numerator and denominator and use the Cauchy-Schwarz inequalities $-\|z\|\|w\| \le \langle z, w\rangle \le \|z\|\|w\|$. Finally, you may use without proof the following standard calculus fact about limits of ratios of polynomials: given $a_1, \ldots, a_m, b_1 \ldots, b_m \in \mathbb{R}$ with $a_m, b_m \neq 0$, one has

$$\lim_{t\to\infty} \frac{\sum_{i=1}^m a_i t^i}{\sum_{i=1}^m b_i t^i} = \frac{a_m}{b_m}.$$

2. Suppose that $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex function and $x \in \operatorname{dom} f$. Show that for any $d \in \mathbb{R}^n$ the function $g_d : (0, \infty) \to (-\infty, +\infty]$ defined by

$$g_d(\alpha) = \frac{f(x + \alpha d) - f(x)}{\alpha}$$

is nondecreasing.

3. Nonconvex Projections (similar to exercise 2.11 in the text): Let $C \subset \mathbb{R}^n$ be a closed but possibly nonconvex set, and consider any point $x \in \mathbb{R}^n$.

   (a) Show that the function $g(w) = \|w - x\|$ must have a nonempty, compact set of minima over C. Denote this set by $P_C(x)$.

   (b) Show that $d_C(x) = \inf_{w\in C} \|w - x\|$ is an everywhere finite-valued and continuous function of $x \in \mathbb{R}^n$.

   (c) Give an example showing that if C is not convex, d_C need not be convex.

4. Given a set $X \subseteq \mathbb{R}^n$, its *indicator function* is the function $\delta_X : \mathbb{R}^n \to (\infty, +\infty]$ given by

$$\delta_X(x) = \begin{cases} 0, & \text{if } x \in X \\ +\infty, & \text{if } x \notin X. \end{cases}$$

   (a) Show that if X is a closed set, δ_X is a closed function.

   (b) Show that if X is a convex set, δ_X is a convex function.

5. Take any closed proper function $f : \mathbb{R}^n \to (\infty, +\infty]$ and scalar $\lambda > 0$. Then the *Moreau regularization* or *Moreau envelope* of f is the function $\hat{f}_\lambda : \mathbb{R}^n$ given by

$$\hat{f}_\lambda(x) = \inf_{w \in \mathbb{R}^n} \left\{ f(w) + \frac{1}{2\lambda} \|w - x\|^2 \right\}.$$

   (a) Show that $\hat{f}_\lambda(x) \leq f(x)$ for all $x \in \mathbb{R}^n$.

   (b) Show that if x^* is a global minimizer of f, then $\hat{f}_\lambda(x^*) = f(x^*)$.

   (c) Show that the properness of f implies that $\hat{f}_\lambda$ can never take the value $+\infty$. Further, show that if f is convex, $\hat{f}_\lambda$ cannot take the value $-\infty$.

   (d) Consider any closed set C. Show that for any $\lambda > 0$, we have

$$d_C(x) = \sqrt{2\lambda (\widehat{\delta_C})_\lambda(x)},$$

   where d_C is the distance function defined in problem 3 and δ_C is the indicator function of C, as defined in problem 4.