Special Topics in Operations Research 16:711:611

Convex Analysis and Optimization

Spring 2009 Rutgers University Prof. Eckstein

Homework 2

Due Wednesday, February 4

1. Prove the following fact, used in the proof of Proposition 1.5.1(b) in the text: Let $x, y \in \mathbb{R}^n$, and let $\{z^k\} \subset \mathbb{R}^n$ be an unbounded sequence; then

$$\lim_{k \to \infty} \frac{\|z^k - x\|}{\|z^k - y\|} = 1.$$

You may use without proof the following standard calculus fact about limits of ratios of polynomials: given $a_1, \ldots, a_m, b_1 \ldots, b_m \in \mathbb{R}$ with $a_m, b_m \neq 0$, one has

$$\lim_{t \to \infty} \frac{\sum_{i=1}^m a_i t^i}{\sum_{i=1}^m b_i t^i} = \frac{a_m}{b_m}.$$

2. Suppose that $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex function and $x \in \operatorname{dom} f$. Show that for any $d \in \mathbb{R}^n$ the function $g_d : (0, \infty) \to (-\infty, +\infty]$ defined by

$$g_d(\alpha) = \frac{f(x + \alpha d) - f(x)}{\alpha}$$

is nondecreasing.

3. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a proper closed convex function, and suppose $x \in \operatorname{dom} f$. Define $r_f : \mathbb{R}^n \to (-\infty, +\infty]$ to be the function whose epigraph is $R_{\operatorname{epi} f}$, the recession cone of $\operatorname{epi} f$. Show for any $d \in \mathbb{R}^n$ that

$$r_f(d) = \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha} = \lim_{\alpha \to \infty} \frac{f(x + \alpha d) - f(x)}{\alpha}.$$

Hint: prove the first equality from the definition of r_f, and then use the result of question 2 to prove the second equality.