Special Topics in Operations Research 16:711:611

Convex Analysis and Optimization

Spring 2009 Rutgers University Prof. Eckstein

Homework 3

Due Wednesday, February 25

1. Problems 3.4(a)-(c) on page 212 of the textbook.
2. *Cone separation.* Suppose $K \subset \mathbb{R}^n$ is a nonempty closed convex cone. Show that if $z \in \mathbb{R}^n$ and $z \notin K$, then there exists $a \in K^*$ with $\langle a, z \rangle > 0$.
3. Show that if $C_1, C_2 \subseteq \mathbb{R}^m$ are closed convex cones, then $(C_1 \cap C_2)^* = \mathrm{cl}(C_1^* + C_2^*)$ (note: this is the main result of problem 3.4(d) in the textbook).
4. Let A be an $m \times n$ real matrix, and $C \subseteq \mathbb{R}^m$ be a closed convex cone. Define
$$K = \{x \in \mathbb{R}^n \mid Ax \in C\} \qquad\qquad P = \{A^\top y \mid y \in C^*\}.$$
   (a) Show that K is a closed convex cone.
   (b) Show that $K^* = \mathrm{cl}\, P$.
   (c) Show that $P^* = K$.
5. A cone K is called *self-dual* if $K^* = -K$. Show that the following cones are self-dual:
   (a) The nonnegative orthant $\{x \in \mathbb{R}^n \mid x \geq 0\}$
   (b) The *Lorentz cone* (also called the "ice cream cone") in $\mathbb{R}^{n+1}$, defined as follows:
$$K = \{(x, w) \in \mathbb{R}^n \times \mathbb{R} \mid w \geq \|x\|\}.$$