Special Topics in Operations Research 16:711:611

Convex Analysis and Optimization

Spring 2009　　Rutgers University　　Prof. Eckstein

Homework 5

Due Wednesday, April 8

1. Consider an optimization problem of the form

$$\begin{array}{lll} \min & f(x) & \\ \text{S.T.} & Ax = b & \\ & h_j(x) \le 0 & j = 1, \ldots, r \\ & x \in X, & \end{array} \tag{1}$$

where

- $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex function
- A is an $m \times n$ matrix and $b \in \mathbb{R}^m$
- For $j = 1, \ldots, r$, $h_j : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function
- X is a convex set.

Let a_i denote row i of A, $i = 1, \ldots, m$, represented as a column vector. Suppose that there exists a point $\overline{x} \in \mathbb{R}^n$ with the following properties:

- $\overline{x} \in \operatorname{ri} \operatorname{dom} f$
- $A\overline{x} = b$
- For $j = 1, \ldots, r$, $h_j(\overline{x}) < 0$
- $x \in \operatorname{ri} X$.

Show that for $x^* \in \mathbb{R}^n$ to be a solution of (1), it is necessary and sufficient that there exist $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^r$ such that

$$\begin{array}{lll} 0 \in \partial f(x^*) + \sum_{i=1}^m \lambda_i^* a_i + \sum_{j=1}^r \mu_j^* \nabla h_j(x^*) + N_X(x^*) & & \sum_{j=1}^r \mu_j^* h_j(x) = 0 \\ Ax^* = b \qquad h_i(x^*) \le 0, \;\; i = 1, \ldots, r & & \mu^* \ge 0. \end{array}$$

You may use results proved in class.

2. (a) Let $K \subseteq \mathbb{R}^m$ be a convex cone. Show that for any $x \in K$,

$$N_K(x) = \{ y \in K^* \mid \langle x, y \rangle = 0 \}$$
$$T_K(x) = \operatorname{cl} \{ z - \alpha x \mid z \in K, \, \alpha \ge 0 \}$$

(b) Suppose A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, and let $Z = \{x \in \mathbb{R}^n \mid Ax - b \in K\}$. Show that, for $x \in Z$,

$$N_Z(x) = \operatorname{cl}\{A^\top \lambda \mid \lambda \in K^*,\ \langle Ax - b, \lambda \rangle = 0\}. \tag{2}$$

For the remainder of this problem, assume that the "cl" operation may be dropped from (2); for an example of a condition guaranteeing this may done, see Proposition 1.5.8 on page 65 of the Bertsekas text.

(c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and consider the problem

$$\begin{array}{ll} \min & f(x) \\ \text{S.T.} & Ax - b \in K \end{array} \tag{3}$$

Show that if $x^* \in \mathbb{R}^n$ is a local minimum for (3), there must exist $\lambda^* \in \mathbb{R}^m$ such that

$$\nabla f(x^*) + A^\top \lambda^* = 0 \qquad \lambda^* \in K^* \qquad \langle Ax^* - b, \lambda^* \rangle = 0.$$

3. For each of the following choices of $f : \mathbb{R} \to (-\infty, +\infty]$, compute the convex conjugate function $\widehat{f}$:

(a) $f(x) = \frac{1}{2}x^2$

(b) For $a, b \in \mathbb{R}$, $a < b$, setting $f = \delta_{[a,b]}$, that is, $f(x) = 0$ whenever $a \le x \le b$, and otherwise $f(x) = +\infty$

(c) $f(x) = e^x$ (the standard exponential function).