Special Topics in Operations Research 16:711:611

Convex Analysis and Optimization

Spring 2009 Rutgers University Prof. Eckstein

Homework 6

Due Wednesday, April 22

1. Given some closed proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and nonempty closed convex cone $K \subseteq \mathbb{R}^n$, consider the conic optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{ST} & x \in K. \end{array} \tag{1}$$

   (a) Using the Fenchel-like duality scheme from the April 8 class, show that the dual problem to (1) may be expressed as

$$\begin{array}{ll} \max & -\widehat{f}(\lambda) \\ \text{ST} & \lambda \in -K^*, \end{array} \tag{2}$$

   Aside: note that the cone $-K^* = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geq 0 \;\; \forall\, x \in K\}$ is called the *dual cone* to K.

   (b) In the setup of question 1a, show that if $\operatorname{ri} \operatorname{dom} f \cap \operatorname{ri} K \neq 0$ and (1) has a finite optimal solution, then strong duality holds between (1) and (2):

$$\min \{f(x) \mid x \in K\} = \max \left\{ -\widehat{f}(\lambda) \mid \lambda \in -K^* \right\}.$$

   (c) Show that if $K = V$, where V is some linear subspace of $\mathbb{R}^n$, then the dual problem (2) takes the form

$$\begin{array}{ll} \max & -\widehat{f}(\lambda) \\ \text{ST} & \lambda \in V^\perp, \end{array} \tag{3}$$

   where $V^\perp$ denotes the orthogonal complement of V.

   (d) Now using the Rockafellar conjugate duality scheme, show that that if we set

$$F(x, u) = \begin{cases} f(x), & \text{if } x - u \in K \\ +\infty, & \text{if } x - u \notin K, \end{cases}$$

   we obtain the same dual problem (2). What is the Lagrangian $L(x, \lambda)$ in this case?

   (e) Show that in the setup of question 1d, the following additional conditions are sufficient to assure strong duality between the problems (1) and (2):

   - There is some $\bar{\lambda} \in -K^*$ with $\widehat{f}(\bar{\lambda}) < \infty$
   - If $\{z^k\} \subset \mathbb{R}^n$ is a sequence with $\|z^k\| \to \infty$, then $f(z^k) \to +\infty$.

   Note: there are many possible alternative sufficient conditions.

2. In this problem, you will show how to recover the Fenchel duality scheme from the Rockafellar conjugate scheme, and the proper form of the Lagrangian for the Fenchel problem. Consider two functions closed proper convex functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, and an $m \times n$ matrix M. Consider the standard Fenchel primal problem

$$\min_{x \in \mathbb{R}^n} f(x) + g(Ax). \tag{4}$$

   (a) Show that if we set $F(x, u) = f(x) + g(Ax + u)$, then F is closed and convex, and we obtain the Lagrangian

   $$L(x, \lambda) = f(x) + \langle Ax, \lambda \rangle - \widehat{g}(\lambda),$$

   with the convention that if both $f(x)$ and $\widehat{g}(\lambda)$ are $+\infty$, then $L(x, \lambda) = +\infty$.

   (b) Show that Rockafellar conjugate duality scheme assigns to (4) the dual problem

   $$\max_{\lambda \in \mathbb{R}^m} -\widehat{f}(-A^\top \lambda) - \widehat{g}(\lambda), \tag{5}$$

   that is, the same dual as you obtain in the Fenchel approach.