Special Topics in Operations Research 16:711:611

Convex Analysis and Optimization

Spring 2009 Rutgers University Prof. Eckstein

Homework 7

Due Wednesday, April 29

1. *Lagrangian Relaxation.* One common use of Lagrangian relaxation/subgradient algorithms is in combinatorial optimization. Suppose we have the problem

$$\begin{array}{ll} \min & c^\top x \\ \text{S.T.} & Ax = b \\ & x \in X, \end{array}$$

where X is a large but finite set of points (hence nonconvex) over which one can easily optimize a linear function. However, with the addition of the constraints $Ax = b$, the optimization task becomes difficult. The Lagrangian relaxation approach attempts to reduce the harder task to a sequence of easier ones via the recursions

$$x^{k+1} \in \text{Arg}\ \min_{x \in X} \left\{ c^\top x + \langle \lambda^k, Ax - b \rangle \right\} \tag{1}$$

$$\lambda^{k+1} = \lambda^k + \alpha_k (Ax^{k+1} - b), \tag{2}$$

where $\alpha_k = \tau_k \gamma_k$ is a subgradient method stepsize. Note that the step (1) reduces to just minimizing the linear objective given by $c + A^\top \lambda$ over X, which is assumed to be "easy".

A classic case of this form is the Held-Karp algorithm for the traveling salesman problem, where X is taken to be the set of all "1-tree" (trees in a graph, plus one additional edge), and the constraints $Ax = b$ express that every node should have degree 2.

Suppose we apply use the standard duality-generating function

$$F(x, u) = \begin{cases} c^\top x, & \text{if } x \in X,\ Ax - b + u = 0 \\ +\infty, & \text{otherwise,} \end{cases}$$

even though X is not convex. Show that the optimal value of the dual function $F^*(0, \lambda)$ one can obtain in this case is at most the value z^* of the linear programming problem

$$\begin{array}{ll} \min & c^\top x \\ \text{S.T.} & Ax = b \\ & x \in \text{conv}\, X, \end{array} \tag{3}$$

and thus that $\min_{x \in X} \{c^\top x + \langle \lambda^k, Ax - b \rangle\}$ cannot exceed z^*. *Hint:* show that $\text{cl}\,\text{conv}\, F$ is equal to the standard duality-generating function of problem (3) above.

2. Refer back to problem 2 of the previous homework, where we use the duality-generating function

$$F(x, u) = f(x) + g(Ax + u)$$

on the problem $\min_{x \in \mathbb{R}^n} f(x) + g(Ax)$. Show that the subgradient method for the corresponding dual problem $\min_{\lambda \in \mathbb{R}^m} \{-F^*(0, \lambda)\}$ can be implemented via the recursions

$$\begin{aligned} x^{k+1} &\in \text{Arg min}_{x \in \mathbb{R}^n} \{f(x) + \langle A^\top \lambda^k, x \rangle\} \\ y^{k+1} &\in \text{Arg min}_{y \in \mathbb{R}^m} \{g(y) - \langle \lambda^k, y \rangle\} \\ \lambda^{k+1} &= \lambda^k + \alpha_k (Ax^{k+1} - y^{k+1}), \end{aligned}$$

where $\alpha_k = \tau_k \gamma_k$ as above.