Nonlinear Optimization

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Homework 5: Optimality Conditions and Lagrange Multipliers

Note that homework assignment 4 was the take-home midterm, so this is assignment 5.

1. Suppose, for $i = 1, \ldots, \ell$, that $C_i \subseteq \mathbb{R}^{p_i}$ are nonempty cones. Show that their Cartesian product $C_1 \times \cdots \times C_\ell$ is a cone and that $(C_1 \times \cdots \times C_\ell)^\circ = C_1^\circ \times \cdots \times C_\ell^\circ$.

2. Take two vectors $a, b \in \mathbb{R}^m$ with $a < b$ (as usual, inequalities between vectors are understood to apply to every pair of corresponding components). Let Y_0 be the "box" set $\{y \in \mathbb{R}^m \mid a \le y \le b\}$.

   (a) For any point $y_0 \in Y_0$, give expressions for the feasible direction cone $K_{Y_0}(y_0)$ and its polar, the normal cone $N_{Y_0}(y_0)$.

   (b) For some continuously differentiable function $g : \mathbb{R}^n \to \mathbb{R}^m$, consider the constraint system $X = \{x \in X_0 \mid g(x) \in Y_0\}$ with $X_0 = \mathbb{R}^n$, that is, let $X = \{x \in \mathbb{R}^n \mid g(x) \in Y_0\}$. Show that if the gradients $\nabla g_1(x_0), \ldots, \nabla g_m(x_0)$ are linearly independent, then the system X is metrically regular at $x_0 \in \mathbb{R}^n$ (note: this simple condition is much stronger than necessary).

   (c) Suppose we know that $\nabla g_1(x), \ldots, \nabla g_m(x)$ are linearly independent for all $x \in \mathbb{R}^n$. Derive a set of necessary local optimality conditions for the optimization problem

   $$\begin{array}{ll} \min & f(x) \\ \text{ST} & g(x) \in Y_0 \end{array} \qquad \text{or equivalently} \qquad \begin{array}{ll} \min & f(x) \\ \text{ST} & a \le g(x) \le b. \end{array}$$

   There should m Lagrange multipliers. You may use without proof that finitely generated cones in $\mathbb{R}^n$ are closed.

3. A *simplex* is a set of the form $S(R) = \{x \in \mathbb{R}^n \mid x \ge 0,\ x_1 + \ldots + x_n = R\}$, for some $R > 0$. Consider the problem of projecting an arbitrary point $y \in \mathbb{R}^n$ onto $S(R)$:

   $$\begin{array}{ll} \min & \frac{1}{2}\|x - y\|^2 \\ \text{ST} & \sum_{i=1}^m x_i = R \\ & x \ge 0 \end{array}$$

   (a) Considering the problem as having $X_0 = \mathbb{R}^n$ and treating the constraints $x \ge 0$ as being of the form $g(x) \le 0$, show that this problem satisifes a constraint qualification.

   (b) Write the Karush-Kuhn-Tucker (KKT) conditions for this problem, using multipliers $\lambda_i \ge 0$, $i = 1, \ldots, n$ for the constraints $x \ge 0$, and a single multiplier μ for the constraint $x_1 + \ldots + x_n = R$. **Note: problem continues overleaf.**

(c) Show that once one chooses μ, the KKT conditions uniquely determine the values of x_i and λ_i, $i = 1, \ldots, n$.

(d) Devise an algorithm to find the correct value of μ by sorting the elements y_i of y and then doing $\mathrm{O}(n)$ additional work ("$\mathrm{O}(n)$" means "bounded by something proportional to n"). *Instructor's note: careful analysis of the complexity of this problem, drawing on the theory of linear-time median finding, can remove the necessity of fully sorting the elements of y, and reduces the entire solution complexity to $\mathrm{O}(n)$; however, this refinement involves techniques beyond the scope of this course.*

(e) Write a MATLAB program implementing your algorithm (MATLAB's `sort` function will sort a vector). Hand in a printout of the `.m`-file of your function and its output x in the following cases:

i. $n = 2, y = (1, 2), R = 2$

ii. $n = 2, y = (-1, 2), R = 2$

iii. $n = 4, y = (-1, 2, 3, 4), R = 5$

iv. $n = 4, y = (-1, 2, 3, 4), R = 3$

v. $n = 4, y = (4, 3, -1, 2), R = 3$

vi. $n = 4, y = (-1, -10, 8, 10), R = 16$