Nonlinear Optimization

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Homework 6: Lagrangian Duality and Related Topics

1. **Duality for quadratic programming.** Assuming that Q is symmetric, solve problem 4.1 on page 203 of the Ruszczyński book. Show that the dual problem can also be expressed as having a quadratic objective and linear constraints.

2. **Generalized convexity.** Let $K \subseteq \mathbb{R}^m$ be a nonempty closed convex cone. A function $g : \mathbb{R}^n \to \mathbb{R}^m$ is called *K-convex* if for all $\alpha \in (0,1)$ and $x, y \in \mathbb{R}^n$, we have

$$g\big(\alpha x + (1-\alpha)y\big) - \big[\alpha g(x) + (1-\alpha)g(y)\big] \in K.$$

   Show the following:

   (a) For $m = 1$, the ordinary definition of convexity of g is equivalent to $\mathbb{R}_-$-convexity, and the ordinary definition of concavity of g is equivalent to $\mathbb{R}_+$-convexity.

   (b) For general $m \geq 1$, g being K-convex implies $\{x \in \mathbb{R}^n \mid g(x) \in K\}$ is a convex set.

   (c) If $g(x) = Ax + b$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, then g is K-convex for any nonempty closed convex cone K.

   (d) If g is K-convex and $\lambda \in K^\circ$, the function $h : x \mapsto \lambda^\top g(x)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is convex in the ordinary sense.

3. **Duality with general conic constraints.** This problem is a generalization of the duality analysis performed in class. Suppose

   - $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable
   - $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable
   - $K \subseteq \mathbb{R}^m$ is a nonempty closed convex cone
   - $X_0 \subseteq \mathbb{R}^n$ is a nonempty closed set.

   Consider the problem

$$\begin{array}{ll} \min & f(x) \\ \text{ST} & g(x) \in K \\ & x \in X_0, \end{array} \tag{1}$$

   (a) Define the Lagrangian of (1) to be

$$L(x, \lambda) = f(x) + \lambda^\top g(x),$$

   and the corresponding primal function to be $L_{\mathrm{P}}(x) = \sup_{\lambda \in K^\circ}\{L(x,\lambda)\}$. Show that

$$L_{\mathrm{P}}(x) = \begin{cases} f(x), & g(x) \in K \\ +\infty, & g(x) \notin K. \end{cases}$$

(b) Next define the dual function to be

$$L_{\mathrm{D}}(\lambda) = \inf_{x \in X_0} \{L(x, \lambda)\}$$

and define $\Lambda_0 = K^\circ$. We define the primal problem to be that of minimizing $L_{\mathrm{P}}(x)$ over $x \in X_0$ and the dual problem to be that of maximizing $L_{\mathrm{D}}(\lambda)$ over $\lambda \in \Lambda_0 = K^\circ$. Note that the results we proved in class about the relationship of the primal and dual problems continue to hold verbatim with exactly the same proofs, since those proofs did not rely on the specific structure of Λ_0 we assumed. In particular, $L_{\mathrm{D}}(x) \leq L_{\mathrm{P}}(\lambda)$ for all $x \in X_0$ and $\lambda \in \Lambda_0 = K^\circ$, and $L_{\mathrm{D}}(x^*) = L_{\mathrm{P}}(\lambda^*)$ if and only if (x^*, λ^*) is a saddle point of the Lagrangian. Show that if X_0 is convex, f is convex, and g is K-convex, then $(x^*, \lambda^*) \in X_0 \times \Lambda_0$ is a saddle point of the Lagrangian if and only if

$$g(x^*) \in K \qquad \langle \lambda^*, g(x^*) \rangle = 0 \qquad 0 \in \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + N_{X_0}(x^*). \tag{2}$$

The analysis may be patterned on the more specific proof given in class for the case $K = \mathbb{R}_+^{m_1} \times \{0\}$ (where the 0 is interpreted as being in $\mathbb{R}^{m_2}$), or the similar proof of Theorem 4.7 in the Ruszczyński textbook.

(c) In the previous part (b), are the convexity of f and the K-convexity of g needed for the "only if" assertion? That is, if we know that $(x^*, \lambda^*) \in X_0 \times K^\circ$ is a saddle point of the Lagrangian, can we assert that (2) holds even if f is not convex or g is not K-convex (or both)?

4. **A simple symmetric form of duality.** Suppose f is a proper function $\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and V is a nonempty linear subspace of $\mathbb{R}^n$. In your analysis, you may assume A to be some $m \times n$ matrix of rank $m \leq n$ such that $V = \{x \in \mathbb{R}^n \mid Ax = 0\}$. Consider the problem

$$\begin{array}{ll} \min & f(x) \\ \mathrm{ST} & x \in V. \end{array} \tag{3}$$

(a) Show that the dual of (3) is equivalent to the problem

$$\begin{array}{ll} \max & -f^*(\lambda) \\ \mathrm{ST} & \lambda \in V^\perp \end{array} \qquad \text{or equivalently} \qquad \begin{array}{ll} \min & f^*(\lambda) \\ \mathrm{ST} & \lambda \in V^\perp, \end{array} \tag{4}$$

where f^* denotes the conjugate function of f and $V^\perp$ is the subspace orthogonal to V.

(b) Show that if f is closed proper convex and has at least one affine minorant, the dual of the dual problem (4), once placed in "min" form, is the same problem as the original problem (3).