Homework 5: Optimality Conditions and Lagrange Multipliers

Note that homework assignment 4 was the take-home midterm, so this is assignment 5.

1. Suppose, for $i = 1, \ldots, \ell$, that $C_i \subseteq \mathbb{R}^{p_i}$ are nonempty cones. Show that their Cartesian product $C_1 \times \cdots \times C_\ell$ is a cone and that $(C_1 \times \cdots \times C_\ell)^\circ = C_1^\circ \times \cdots \times C_\ell^\circ$.

2. Take two vectors $a, b \in \mathbb{R}^m$ with $a < b$ (as usual, inequalities between vectors are understood to apply to every pair of corresponding components). Let $Y_0$ be the “box” set $\{y \in \mathbb{R}^m \mid a \leq y \leq b\}$.

   (a) For any point $y_0 \in Y_0$, give expressions for the feasible direction cone $K_{Y_0}(y_0)$ and its polar, the normal cone $N_{Y_0}(y_0)$.

   (b) For some continuously differentiable function $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$, consider the constraint system $X = \{x \in X_0 \mid g(x) \in Y_0\}$ with $X_0 = \mathbb{R}^n$, that is, let $X = \{x \in \mathbb{R}^n \mid g(x) \in Y_0\}$. Show that if the gradients $\nabla g_1(x_0), \ldots, \nabla g_m(x_0)$ are linearly independent, then the system $X$ is metrically regular at $x_0 \in \mathbb{R}^n$ (note: this simple condition is much stronger than necessary).

   (c) Suppose we know that $\nabla g_1(x), \ldots, \nabla g_m(x)$ are linearly independent for all $x \in \mathbb{R}^n$. Derive a set of necessary local optimality conditions for the optimization problem

   $$\min_{ST \ g(x) \in Y_0} f(x) \quad \text{or equivalently} \quad \min_{ST \ a \leq g(x) \leq b} f(x).$$

   There should $m$ Lagrange multipliers. You may use without proof that finitely generated cones in $\mathbb{R}^n$ are closed.

3. A simplex is a set of the form $S(R) = \{x \in \mathbb{R}^n \mid x \geq 0, \ x_1 + \ldots + x_n = R\}$, for some $R > 0$. Consider the problem of projecting an arbitrary point $y \in \mathbb{R}^n$ onto $S(R)$:

   $$\min_{ST} \frac{1}{2} \|x - y\|^2 \quad \text{subject to} \quad \sum_{i=1}^{m} x_i = R \quad \text{and} \quad x \geq 0$$

   (a) Considering the problem as having $X_0 = \mathbb{R}^n$ and treating the constraints $x \geq 0$ as being of the form $g(x) \leq 0$, show that this problem satisfies a constraint qualification.

   (b) Write the Karush-Kuhn-Tucker (KKT) conditions for this problem, using multipliers $\lambda_i \geq 0, \ i = 1, \ldots, n$ for the constraints $x \geq 0$, and a single multiplier $\mu$ for the constraint $x_1 + \ldots + x_n = R$. 
(c) Show that once one chooses \( \mu \), the KKT conditions uniquely determine the values of \( x_i \) and \( \lambda_i, i = 1, \ldots, n \).

(d) Devise an algorithm to find the correct value of \( \mu \) by sorting the elements \( y_i \) of \( y \) and then doing \( O(n) \) additional work ("\( O(n) \)" means "bounded by something proportional to \( n \)"). Instructor’s note: careful analysis of the complexity of this problem, drawing on the theory of linear-time median finding, can remove the necessity of fully sorting the elements of \( y \), and reduces the entire solution complexity to \( O(n) \); however, this refinement involves techniques beyond the scope of this course.

(e) Write a MATLAB program implementing your algorithm (MATLAB’s \texttt{sort} function will sort a vector). Hand in a printout of the .m-file of your function and its output \( x \) in the following cases:

- i. \( n = 2, y = (1, 2), R = 2 \)
- ii. \( n = 2, y = (-1, 2), R = 2 \)
- iii. \( n = 4, y = (-1, 2, 3, 4), R = 5 \)
- iv. \( n = 4, y = (-1, 2, 3, 4), R = 3 \)
- v. \( n = 4, y = (4, 3, -1, 2), R = 3 \)
- vi. \( n = 4, y = (-1, -10, 8, 10), R = 16 \)