Nonlinear Optimization
Fall 2019, Rutgers University
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Homework 6: Duality and Related Topics

1. **Duality for quadratic programming.** Assuming that $Q$ is symmetric, solve problem 4.1 on page 203 of the Ruszczyński book. Show that the dual problem can also be expressed as having a quadratic objective and linear constraints.

2. **Fenchel duality from Lagrangian duality** Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be closed proper convex functions, and $M$ be any $m \times n$ matrix. Define a “Lagrangian” function $L$ on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$L(x, \lambda) = f(x) + \lambda^\top Mx - g^*(\lambda).$$

Set $X_0 = \mathbb{R}^n$ and $\Lambda_0 = \mathbb{R}^m$.

(a) Show that the corresponding primal function $L_P$ is given by

$$L_P(x) = f(x) + g(Mx).$$

Thus, the primal problem corresponding to $L$ is to minimize $f(x) + g(Mx)$.

(b) Show that the corresponding dual function $L_D$ is given by

$$L_D = -f^*(-M^\top \lambda) - g^*(\lambda).$$

Thus, the dual problem from $L$ is to maximize $L_D = -f^*(-M^\top \lambda) - g^*(\lambda)$.

3. **A simple exercise in conjugate functions.** Let $V$ be a linear subspace of $\mathbb{R}^n$ and consider the convex function

$$\delta_V(x) = \begin{cases} 0, & \text{if } x \in V \\ +\infty, & \text{if } x \notin V. \end{cases}$$

Show that

$$\delta_V^*(y) = \begin{cases} 0, & \text{if } y \in V^\perp \\ +\infty, & \text{if } y \notin V^\perp, \end{cases}$$

where $V^\perp$ denotes the orthogonal complement of $V$.

4. **A simple symmetric form of duality.** Suppose $f$ is a closed proper convex function $\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $V$ is a nonempty linear subspace of $\mathbb{R}^n$. Consider the problem

$$\min_{x \in V} f(x).$$

By setting $n = m$, $M = I$, and $g = \delta_V$ in problem 2 and also using problem 3, show that a dual problem to (1) may be written as

$$\max_{\lambda \in V^\perp} -f^*(\lambda) \quad \text{or equivalently} \quad \min_{\lambda \in V^\perp} f^*(\lambda).$$

(2)